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A simultaneous expansion for the electromagnetic field of a relativistic charged particle

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Abstract. The expressions for electromagnetic potentials and fields of a charged particle in arbitrary motion in terms of simultaneous characteristics of the motion have been derived with the help of the Lagrange expansion for a retarded function. The expressions do not contain expansions in powers of the ratio v/c, and hence they are effective for particles with arbitrarily high velocities. The derived expressions may be useful in classical field theory and may serve as a starting point for development of a non-quantal relativistic statistical mechanics.

1. Introduction

Due to the finite speed of propagation the field of a charged particle is defined by its position, velocity and acceleration at a previous moment of time, ie in a system of many particles different 'times' should be used for different particles. This means it is impossible to describe a state of a system of charged particles in a phase space of their coordinates and momenta, so that one should introduce independent degrees of freedom for the electromagnetic field. Therefore the equations of motion for the particles should be supplemented with the equations of the field (quantized) which makes consideration of electromagnetic systems rather complicated. It is particularly difficult to develop a relativistic statistical theory when one needs a common time for all particles to describe the evolution of the system.

However, as is shown in this paper, the electromagnetic field of a particle can be expressed in terms of its position, velocity and time derivatives of the velocity at simultaneous and not at previous moments of time. It is not necessary then to introduce independent degrees of freedom for the field, and states of the system can be completely described in a phase space of coordinates, momenta and time derivatives of the momenta (we call them 'accelerations') of the particles only. Such a description is especially appropriate for statistical or kinetic theories because it allows one to use a common laboratory time for all particles.

Another advantage of this approach is the possibility of restricting oneself to a numerable set of variables for a system of interacting particles. When the field is described independently, one should deal with a continuum of the field variables (say, potentials defined at each point of space), and it is quantization that usually results in a numerable set of the field oscillations. The proposed approach in a sense can be an alternative

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to the quantum one. Although it is not clear how these two methods of description are interrelated, the non-quantal relativistic theory itself seems to be justified from theoretical as well as from practical points of view (Schwinger 1949, Hakim 1967).

The value of the expressions derived below is due to the fact that they do not contain expansions in powers of the ratio v/c, where v and c are speeds of particle and light respectively. Therefore, these expressions are very general and can be used for a particle travelling arbitrarily fast even when its velocity is close to that of light (ultra-relativistic case). These expressions contain only series, in powers of accelerations of different orders.

Since the expressions for potentials and fields include the universal laboratory time common for all particles, they are not manifestly covariant. However, they result from expansions of solutions to the covariant Maxwell equations, and this implies their actual independence of the choice of the frame of reference. Since, in statistical consideration, time plays a very important role due to irreversibility of processes in a system, there is no need to insist on the covariant form of the theory where time is just one of the four coordinates. Some loss of mathematical elegance is compensated for, in the author's opinion, by the physical clarity and simplicity of the expressions derived. Moreover, it is possible, when desirable, to rewrite these expressions in explicit covariant form as illustrated in § 3 for the case of the radiative reaction force.

2. The simultaneous expansions for retarded potentials and fields

The electromagnetic field of a charge e in arbitrary motion is defined at a point with coordinates $\boldsymbol{x}$ at a moment of time t by the following Liénard–Wiechert potentials:

$$\phi(\boldsymbol{x}, t) = \frac{e}{R'(1-\boldsymbol{n}' \cdot \boldsymbol{\beta}')}, \qquad \boldsymbol{A}(\boldsymbol{x}, t) = \frac{e\boldsymbol{\beta}'}{R'(1-\boldsymbol{n}' \cdot \boldsymbol{\beta}')}. \tag{1}$$

Here, R' is the magnitude of the vector $\boldsymbol{R}' = \boldsymbol{x}-\boldsymbol{x}(t')$ from the charge to the point of observation $\boldsymbol{x}$ at a previous moment of time t', $\boldsymbol{x}(t')$ being the coordinates of the charge at that moment; $\boldsymbol{n}' = \boldsymbol{R}'/R'$ is a unit vector along $\boldsymbol{R}' = \boldsymbol{R}(t')$, and $\boldsymbol{\beta}' = \boldsymbol{v}(t')/c$ is the ratio of the velocity of the charge to the speed of light. The moment of time t' is defined by the causality condition

$$t' = t-\frac{R(t')}{c}. \tag{2}$$

The corresponding fields $\boldsymbol{E}$ and $\boldsymbol{H}$ can be found by differentiation of ϕ and $\boldsymbol{A}$ with respect to the coordinates x, y, z, of the point of observation $\boldsymbol{x}$ and the time of observation t according to the relations

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}-\nabla\phi, \qquad \boldsymbol{H} = \nabla\times\boldsymbol{A}. \tag{3}$$

These differentiations are not straightforward because ϕ and $\boldsymbol{A}$ depend on t' according to (1), and their implicit dependence on x, y, z, t, is defined by the condition (2).

The calculation of the previous position and velocity of the particle is a difficult problem for an arbitrary law of motion $\boldsymbol{x}(t)$, because the condition (2) contains the unknown distance $R(t')$. Fortunately, there exists an expansion (first obtained by Lagrange) that permits one to express a retarded function as an infinite series, all terms of which are defined at simultaneous and not at previous moments of time. For an arbitrary function $u(t')$ of the retarded time t' such an expression has a form (eg Smart 1953 or Wintner 1947)

$$u(t') = u(t) + \sum_{k=1}^{\infty} \frac{(-1)^k}{c^k k!} \frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}\left(R^k \frac{\mathrm{d}u}{\mathrm{d}t}\right), \tag{4}$$

where now in the right-hand side $R = |\boldsymbol{x} - \boldsymbol{x}(t)|$ is the simultaneous separation of the particle from the point of observation.

The series in (4) can be rewritten in the form:

$$u(t') = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[\left(\frac{R}{c}\right)^k u\left(1 - \frac{\boldsymbol{n} \cdot \boldsymbol{v}}{c}\right)\right], \tag{5}$$

if one takes into account that

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -\boldsymbol{n} \cdot \boldsymbol{v}, \tag{6}$$

$\boldsymbol{n} = \boldsymbol{R}/R$ being the unit vector from the particle to the point of observation, and $\boldsymbol{v}$ the velocity of the particle, both simultaneous with the observation.

To simplify the formulae we shall use a special system of units where the speed of light $c = 1$, or introduce instead of time t the relativistic 'time coordinate' $x_0 = ct$. In these units the Liénard–Wiechert potentials (1) can be written with the aid of Lagrange's expansion (5) in the form:

$$\phi = e \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\mathrm{d}^k R^{k-1}}{\mathrm{d}t^k}, \qquad \boldsymbol{A} = e \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\mathrm{d}^k (R^{k-1}\boldsymbol{v})}{\mathrm{d}t^k}. \tag{7}$$

The corresponding expressions for fields can be found from (7) by means of the differentiations (3):

$$\begin{aligned} \boldsymbol{E} &= e \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\mathrm{d}^k}{\mathrm{d}t^k}\{R^{k-2}[\boldsymbol{n} - k(\boldsymbol{n} \cdot \boldsymbol{v})]\}, \\ \boldsymbol{H} &= e \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\mathrm{d}^k}{\mathrm{d}t^k}[R^{k-2}(k-1)(\boldsymbol{n} \times \boldsymbol{v})]. \end{aligned} \tag{8}$$

With the help of the Leibnitz formula for the kth derivative of the product of two functions one can present the expression for the vector potential $\boldsymbol{A}$ in (7) as a double sum:

$$\boldsymbol{A} = e \sum_{m,k=0}^{\infty} \frac{(-1)^{m+k}}{m!k!} \overset{k}{\boldsymbol{v}} \frac{\mathrm{d}^m R^{m+k-1}}{\mathrm{d}t^m},$$

where $\overset{k}{\boldsymbol{v}} = \mathrm{d}^k \boldsymbol{v}/\mathrm{d}t^k$ is the 'acceleration' of the particle of the kth order.

If we define now the 'scalar potential of the kth order' ϕ_k by the following relation

$$\phi_k = e \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\mathrm{d}^m R^{m+k-1}}{\mathrm{d}t^m}, \tag{9}$$

we can express the electromagnetic potentials in terms of these ϕ_k:

$$\phi = \phi_0, \qquad \boldsymbol{A} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \overset{k}{\boldsymbol{v}} \phi_k. \tag{10}$$

The analogous transformation of the expressions (8) for fields permits one to present them in the following form:

$$\begin{aligned} \boldsymbol{E} &= e \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}(k-1)C_{k-1}\overset{k}{\boldsymbol{R}}, \\ \boldsymbol{H} &= e \sum_{k,m=0}^{\infty} \frac{(-1)^{k+m}}{k!\,m!} mC_{k+m-2}\overset{k}{\boldsymbol{R}} \times \overset{m}{\boldsymbol{R}}, \end{aligned} \tag{11}$$

where

$$C_k = \sum_{i=0}^{\infty} \frac{(-1)^i(i+k)}{i!} \frac{\mathrm{d}^i R^{i+k-2}}{\mathrm{d}t^i}, \tag{12}$$

and the index above a letter denotes the derivative with respect to 'time' ct of the corresponding order, eg

$$\dot{\boldsymbol{R}} = \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = -\boldsymbol{v}, \qquad \ddot{\boldsymbol{R}} = -\dot{\boldsymbol{v}}, \ldots, \qquad \overset{k}{\boldsymbol{R}} = \frac{\mathrm{d}^k\boldsymbol{R}}{\mathrm{d}t^k} = -\overset{k-1}{\boldsymbol{v}},$$

since $\boldsymbol{R} = \boldsymbol{x} - \boldsymbol{x}(t)$.

The coefficients ϕ_k and C_k in expansions (10) and (11) for potentials and fields contain accelerations of the particle according to (9) and (12), since the separation of the charge from the observation point $R = |\boldsymbol{x} - \boldsymbol{x}(t)|$ depends on time through the variable vector $\boldsymbol{x}(t)$ which characterizes the motion of the particle. With the help of the expression for the mth derivative of a composite function (eg Gradshteyn and Ryzhik 1965) one can rewrite the expression (9) for ϕ_k in the following form:

$$\phi_k = e \sum_{m=0}^{\infty} \sum \frac{(-1)^{m+p}}{i!\,j!\ldots h!} \frac{\partial^p R^{m+k-1}}{\partial \boldsymbol{R}^p} \boldsymbol{v}^i \left(\frac{\dot{\boldsymbol{v}}}{2!}\right)^j \cdots \left(\frac{\overset{q-1}{\boldsymbol{v}}}{q!}\right)^h, \tag{13}$$

where the second sum is to be performed over all positive integers $k, j, \ldots, h$ that satisfy the equations:

$$i + 2j + \cdots + qh = m \tag{14}$$

$$i + j + \ldots + h = p. \tag{15}$$

To simplify the formulae we have introduced in (13) the following notations for tensors of corresponding ranks in the usual three-dimensional coordinate space:

$$\boldsymbol{R}^p = \underbrace{R_a R_b \ldots R_c}_{p \text{ times}}, \qquad (\dot{\boldsymbol{v}})^j = \underbrace{\dot{v}_a \dot{v}_b \ldots \dot{v}_c}_{j \text{ times}}, \qquad a, b, \ldots, c = 1, 2, 3.$$

For instance, $\partial^p R^{m+k-1}/\partial \boldsymbol{R}^p$ is a tensor of rank p, and the condition (15) makes ϕ_k in (13) scalar quantities as they should be. It is evident that the same transformation can be performed on the expression (12) as well.

We have almost achieved our purpose now, but the expression (13) suffers in a sense because it includes an expansion in powers of $\boldsymbol{v}$ (ie in powers of the ratio v/c). This is undesirable if one prefers not to restrict oneself to the case of a relatively low speed of the particle that originates the field. We want to have a theory appropriate for arbitrarily fast particles even when $v \to 1$ ($v \to c$ in usual units). This is especially important for statistical and kinetic theories where no restrictions on the *possible* magnitudes of velocities can be imposed.

It is possible to find more compact expressions, containing no expansions in powers of $\boldsymbol{v}$, if we make use of the fact that, for a particle in uniform motion, its field can be expressed in terms of its simultaneous, and not retarded, position and velocity. The corresponding expression for the scalar potential is (eg Jackson 1963):

$$\phi = \frac{e}{R[1-v^2+(\boldsymbol{n}\cdot\boldsymbol{v})^2]^{1/2}}, \tag{16}$$

where the unit vector of direction $\boldsymbol{n}$ is the same as in (5) and it is taken at the moment of observation.

The corresponding transformations are indirect and rather lengthy, and they are given in the appendix. The result is:

$$\phi_k = e \sum_{m=0}^{\infty} (-1)^m R^{m+k-1} B_k^m, \tag{17}$$

where

$$B_k^0 = \Phi_{k-1}, \qquad B_k^1 = \frac{\partial \Phi_k}{\partial \boldsymbol{v}} \frac{\dot{\boldsymbol{v}}}{2!}, \ldots$$

$$B_k^m = \sum \frac{1}{i!j!\ldots h!} \frac{\partial^s \Phi_{k+m-1}}{\partial \boldsymbol{v}^s} \left(\frac{\dot{\boldsymbol{v}}}{2!}\right)^i \left(\frac{\ddot{\boldsymbol{v}}}{3!}\right)^j \cdots \left[\frac{\overset{q}{\boldsymbol{v}}}{(q+1)!}\right]^h \qquad (m \geqslant 1) \tag{18}$$

$$i+2j+\ldots+qh = m \tag{19}$$

$$i+j+\ldots+h = s. \tag{20}$$

Here, the scalar functions of coordinates and velocity of the particle $\Phi_k(\boldsymbol{n}, \boldsymbol{v})$ are:

$$\Phi_k = \frac{\boldsymbol{n}^k}{\gamma^2 k!} \frac{\partial^k \Phi_0}{\partial \boldsymbol{v}^k} = \frac{\gamma^{k+2}[z+(1+z^2)^{1/2}]^{k+1}}{(1+z^2)^{1/2}}, \tag{21}$$

where we use the relativistic notation $\gamma = (1-v^2)^{-1/2}$ and put $z = \gamma(\boldsymbol{n}\cdot\boldsymbol{v}) = \boldsymbol{n}\cdot\boldsymbol{u}$, $\boldsymbol{u}$ being the spatial components of the four-velocity of the particle.

The expression (17) shows that ϕ_k are infinite series in increasing powers of the distance R. It is seen from (19) that the total order of time derivatives of the velocity ('accelerations') in each term B_k^m of the series is m, while (20) provides the scalar character of these B_k^m (cf (13)).

Substitution from (17) into (10) gives finally the following expressions for the electromagnetic potentials:

$$\phi = e \sum_{m=0}^{\infty} (-1)^m R^{m-1} B_0^m, \qquad \boldsymbol{A} = e \sum_{m=0}^{\infty} (-1)^m R^{m-1} \boldsymbol{a}_m,$$

$$\boldsymbol{a}_m = \sum_{k=0}^{m} B_k^{m-k} \frac{\overset{k}{\boldsymbol{v}}}{k!}, \tag{22}$$

with B_k^m from (18). The total order of time derivatives of the velocity $\boldsymbol{v}$ in $\boldsymbol{a}_m$ is also m.

The analogous consideration of the expressions (11) for fields yields in the following formulae:

$$\boldsymbol{E} = e \sum_{j=0}^{\infty} (-1)^j \boldsymbol{E}_j, \qquad \boldsymbol{H} = e \sum_{j=0}^{\infty} (-1)^j \boldsymbol{H}_j,$$

where

$$\boldsymbol{E}_j = -C_j^{-1}\boldsymbol{R} + \sum_{i=0}^{j} \frac{j-i}{(j-i+1)!} C_i^{j-i} \overset{j-i}{\boldsymbol{v}}, \qquad \boldsymbol{H}_j = \sum_{i=0}^{j} \boldsymbol{H}_i^j,$$

$$\boldsymbol{H}_i^j = C_i^{j-i-1} \frac{\boldsymbol{R} \times \overset{j-1}{\boldsymbol{v}}}{(j-i)!} + C_i^{j-i} \sum_{k=0}^{\mathrm{E}[(j-i)/2]} \frac{(j-i-2k)\overset{k}{\boldsymbol{v}} \times \overset{j-i-k}{\boldsymbol{v}}}{(k+1)!(j-i-k+1)!}, \tag{23}$$

$\mathrm{E}[(j-i)/2]$ being the integral part of $(j-i)/2$. Here, the scalar functions C_i^j are connected with C_k from (12) in the following way:

$$C_k = \sum_{j=0}^{\infty} (-1)^j C_j^k.$$

They can be presented in the following general form:

$$C_0^k = \Omega_k, \qquad C_l^k = \sum \frac{1}{i!j!\dots h!} \frac{\partial^s \Omega_{k+l}}{\partial \boldsymbol{v}^s} \left(\frac{\dot{\boldsymbol{v}}}{2!}\right)^i \left(\frac{\ddot{\boldsymbol{v}}}{3!}\right)^j \cdots \left(\frac{\overset{q}{\boldsymbol{v}}}{(q+1)!}\right)^h, \tag{24}$$

that is analogous to (18) to the same restrictions (19) and (20) on the indices of the summation $i, j, \dots, h$. The expressions for the scalar functions $\Omega_k(\boldsymbol{R}, \boldsymbol{v})$ can be found by analogy with (21). They are

$$\Omega_k = R^{k-2}\gamma^{k+2} \frac{[z + (1+z^2)^{1/2}]^k [k(1+z^2)^{1/2} - z]}{(1+z^2)^{3/2}}. \tag{25}$$

It is seen from (23), (24) and (25) that

$$\{\boldsymbol{E}_i, \boldsymbol{H}_i\} \sim R^{i-2},$$

and that the total order of time derivatives of the velocity in $\boldsymbol{E}_i, \boldsymbol{H}_i$ is i. Note that the partial differentiation with respect to $\boldsymbol{v}$ in (18) and (24) does not influence dependence on R because it is to be performed at $\boldsymbol{R}$ = constant.

3. Discussion

The expressions (22) and (23) for potentials and fields of the particle respectively give the desirable expansions in terms of simultaneous characteristics of its motion. They imply no restrictions on the magnitude of the velocity of the particle, and they can be used in a non-quantal relativistic statistical mechanics in the form of Klimontovich (1960) and Hakim (1967)—that will be done elsewhere.

We now discuss some applications of these formulae to classical electrodynamics. First of all, it is seen from (23) that the terms $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ corresponding to $j = 0$ do not contain accelerations at all, and their dependence on the distance is R^{-2}. In explicit form they are

$$\boldsymbol{E}_0 = -eC_0^{-1}\boldsymbol{R}, \qquad \boldsymbol{H}_0 = eC_0^{-1}\boldsymbol{R} \times \boldsymbol{v}, \tag{26}$$

and the substitution for $C_0^{-1} = \Omega_{-1}$ from (25) according to (24) gives

$$\boldsymbol{E}_0 = \frac{e\gamma\boldsymbol{n}}{R^2(1+z^2)^{3/2}}, \qquad \boldsymbol{H}_0 = \boldsymbol{v} \times \boldsymbol{E}_0 = \frac{e\gamma\boldsymbol{v} \times \boldsymbol{n}}{R^2(1+z^2)^{3/2}}. \tag{27}$$

These are nothing but the fields of a charge in uniform motion (eg Jackson 1963).

These formulae also permit one to calculate the force that the particle exerts on itself. To find it, one should take the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ at $R = 0$, and substitute the corresponding expressions into the right-hand side of the Lorentz equations of motion of the particle

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m\boldsymbol{v}}{(1-v^2)^{1/2}}\right) = e(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{H}). \tag{28}$$

In the series (23) the terms with $j = 0, 1$ diverge when $R \to 0$ because they contain R^{-2} and R^{-1} respectively. They correspond to the divergent self-energy of the particle and have to be eliminated with the help of an appropriate renormalization of mass. The terms $\boldsymbol{E}_2$ and $\boldsymbol{H}_2$ do not contain R at all (they are proportional to R^0) and remain finite when $R \to 0$. Making use of (23), it is easy to see that they are

$$\begin{aligned} \boldsymbol{E}_2 &= e\left(-C_2^{-1}\boldsymbol{R}+C_1^1\frac{\dot{\boldsymbol{v}}}{2!}+2C_0^2\frac{\ddot{\boldsymbol{v}}}{3!}\right), \\ \boldsymbol{H}_2 &= e\left(-C_2^{-1}\boldsymbol{v}\times\boldsymbol{R}+C_1^0\boldsymbol{R}\times\dot{\boldsymbol{v}}+C_1^1\frac{\boldsymbol{v}\times\dot{\boldsymbol{v}}}{2!}+2C_0^2\frac{\boldsymbol{v}\times\ddot{\boldsymbol{v}}}{3!}+C_0^1\frac{\boldsymbol{R}\times\ddot{\boldsymbol{v}}}{2!}\right). \end{aligned} \tag{29}$$

The other terms in the series with $j > 2$ contain positive powers of R and vanish when $R \to 0$ (at the centre of the particle).

The unit vector $\boldsymbol{n}$ which is the ratio $\boldsymbol{R}/R$ becomes indefinite at the limit $R \to 0$. The simplest possible assumption about it is that it is zero for an isotropic particle. Thus if we omit all terms in (29) that contain $\boldsymbol{R}$ (ie $\boldsymbol{n}$), we get

$$\begin{aligned} \boldsymbol{E}_2(R \to 0) &= e(\tfrac{1}{2}C_1^1\dot{\boldsymbol{v}}+\tfrac{1}{3}C_0^2\ddot{\boldsymbol{v}}), \\ \boldsymbol{H}_2(R \to 0) &= e(\tfrac{1}{2}C_1^1\boldsymbol{v}\times\dot{\boldsymbol{v}}+\tfrac{1}{3}C_0^2\boldsymbol{v}\times\ddot{\boldsymbol{v}}) = \boldsymbol{v}\times\boldsymbol{E}_2. \end{aligned} \tag{30}$$

It follows from (24) that

$$C_1^1 = \frac{1}{2}\frac{\partial\Omega_2}{\partial\boldsymbol{v}}\dot{\boldsymbol{v}}, \qquad C_0^2 = \Omega_2,$$

so that the field can be expressed in terms of the function Ω_2 only which according to (25) is

$$\Omega_2(R \to 0) = \frac{\gamma^4[z+(1+z^2)^{1/2}]^2[2(1+z^2)^{1/2}-z]}{(1+z^2)^{3/2}} = 2\gamma^4,$$

where we again put $z = \gamma(\boldsymbol{n}\,.\,\boldsymbol{v}) = 0$ because of our assumption that $\boldsymbol{n} = 0$ when $R \to 0$.

Substitution into (30) yields in the following final expression for the field:

$$\boldsymbol{E}_2 = \frac{2e}{3}\gamma^4[\ddot{\boldsymbol{v}}+3\gamma^2\dot{\boldsymbol{v}}(\dot{\boldsymbol{v}}\,.\,\boldsymbol{v})], \qquad \boldsymbol{H}_2 = \boldsymbol{v}\times\boldsymbol{E}_2, \tag{31}$$

and for the Lorentz force of the self-action

$$\boldsymbol{F} = \frac{2e^2}{3}\gamma^4[\ddot{\boldsymbol{v}}+\boldsymbol{v}\times(\boldsymbol{v}\times\ddot{\boldsymbol{v}})]+2e^2\gamma^6(\boldsymbol{v}\,.\,\dot{\boldsymbol{v}})[\dot{\boldsymbol{v}}+\boldsymbol{v}\times(\boldsymbol{v}\times\dot{\boldsymbol{v}})]. \tag{32}$$

The expression (32) coincides with the spatial components of the four-vector of the radiation reaction in classical relativistic electrodynamics and it can easily be written in a covariant form. Neither the result itself nor the method of derivation is new, but usually in this way one arrives at an approximate expression for the reaction (eg Page and Adams 1931), and not at the exact relativistic formula (32).

The general expressions (23) are also useful when considering a system of two interacting particles. The two-body problem has no exact solution in the relativistic case because of the infinite number of degrees of freedom of the radiated electromagnetic field. Therefore, in considering such a system one usually has to resort to some approximations. However, unlike the usual approaches, we are now interested in the case of arbitrarily fast particles and cannot exploit the smallness of the ratio v/c and perform expansions in powers of this parameter. This is important, for instance, in the relativistic generalization of Boltzmann's collision integral.

Fortunately, there exists another parameter, proportional to e^2, which allows one to use expansions in series without any restriction on the magnitude of the velocity. Indeed, the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ of a particle are proportional to its charge e. On the other hand, according to the equations of motion (28), acceleration is defined by the Lorentz force which is also proportional to the charge. Therefore, the acceleration (of any 'order') contains a small parameter $e^2/mc^2R_{\min}$, where $R_{\min}$ is the minimum separation of the interacting particles. For the possibility to restrict oneself to the non-quantal theory this separation should exceed the Compton wavelength $\hbar/mc$. This means that in a non-quantal theory this parameter is very small

$$\alpha = \frac{e^2}{mc^2R_{\min}} < \frac{e^2}{mc^2}\frac{mc}{\hbar} = \frac{e^2}{\hbar c} = \frac{1}{137} \ll 1.$$

In the zeroth approximation in this parameter, one has to omit in the expansion (23) all the terms containing accelerations since they are of the first order in α. In this approximation the field of the particle coincides with the field (27) of the charge in uniform motion with the same velocity, though the acceleration may not be zero but just does not influence the field.

In the first approximation one should keep in the expansion (23) only those terms which are linear in accelerations of all orders and ignore their products. This yields in considerably simplified forms the formulae (24). Moreover, the accelerations in these expressions should be calculated by means of successive differentiations of the equations of motion (28) with respect to time, but now in the right-hand side of these equations the fields $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ should be used, because the terms with $j > 0$ contain accelerations and are of higher order in the parameter α.

The radiative reaction force (32) should be taken into account only in the second (and higher) approximation in the parameter α.

Thus one can develop a classical perturbation theory that takes the influence of the radiation field on the motion of the particles into account, but does not introduce additional degrees of freedom for it.

4. Conclusion

The formulae (22) together with (18), and (23) together with (24) and (25), give the desirable expressions for the field of a particle in arbitrary motion in terms of its simultaneous characteristics. These expressions contain time derivatives of all orders of velocity $\boldsymbol{v}$ of the particle, so that the state of the particle together with its field is to be specified in a phase space of an infinite but numerable set of variables $\{\boldsymbol{x}, \boldsymbol{v}, \dot{\boldsymbol{v}}, \ldots, \overset{n}{\boldsymbol{v}}, \ldots\}$.

These expressions being written in a non-covariant form nevertheless have 'correct' properties under Lorentz transformations. They can be written, if desirable, in terms of the covariant derivatives of the four-velocity of the particle with respect to its proper time.

Since we deliberately restrict the consideration to the limits of the classical electrodynamics, all the known difficulties of the theory are present here. In particular, this approach gives nothing new with respect to the divergence of the self-energy of a charged particle.

However, in the author's opinion, this consideration is justified by the possibility to develop a relativistic statistical mechanics with the help of the expressions derived for the fields. These expressions may also be helpful in the classical electrodynamics and even for 'quantization' of electromagnetic field. They may be used as a starting point for interesting speculations about the equations of motion of a charged particle alternative to the Wheeler and Feynman (1945, 1949) theory.

Appendix. Transformation of the expression (13) for ϕ_k

According to the general idea to avoid expansions in powers of $\boldsymbol{v}$, let us isolate the corresponding series in the expressions (13) for ϕ_k. The first terms in the sum (13) are

$$\phi_k = e\left(\sum_{m=0}^{\infty}(-1)^{2m}\frac{\partial^m R^{m+k-1}}{\partial \boldsymbol{R}^m}\frac{\boldsymbol{v}^m}{m!}\right.$$
$$\left.+\sum_{m=2}^{\infty}(-1)^{2m-1}\frac{\partial^{m-1}R^{m+k-1}}{\partial \boldsymbol{R}^{m-1}}\frac{\boldsymbol{v}^{m-2}\dot{\boldsymbol{v}}}{2!(m-2)!}+\ldots\right). \tag{A.1}$$

A rearrangement of the indices of the summations permits one to present (A.1) in the following form:

$$\phi_k = e\left(\psi_{k-1}-\frac{\partial\psi_{k+1}}{\partial \boldsymbol{R}}\frac{\dot{\boldsymbol{v}}}{2!}+\frac{\partial\psi_{k+2}}{\partial \boldsymbol{R}}\frac{\ddot{\boldsymbol{v}}}{3!}+\frac{\partial^2\psi_{k+3}}{\partial \boldsymbol{R}^2}\left(\frac{\dot{\boldsymbol{v}}}{2!}\right)^2\frac{1}{2!}-\ldots\right),$$

where we have introduced a new notation

$$\psi_k(\boldsymbol{R}, \boldsymbol{v}) = \sum_{m=0}^{\infty}\frac{\partial^m R^{m+k}}{\partial \boldsymbol{R}^m}\frac{\boldsymbol{v}^m}{m!}. \tag{A.2}$$

The scalar quantities $\psi_k(\boldsymbol{R}, \boldsymbol{v})$, introduced in (A.2), are functions of the vector $\boldsymbol{R} = \boldsymbol{x}-\boldsymbol{x}(t)$ from the charge to the point of observation, and the vector of velocity $\boldsymbol{v}(t)$ of the particle. To find compact expressions for them, let us consider the field of a charge in uniform motion when

$$\dot{\boldsymbol{v}} = \ddot{\boldsymbol{v}} = \ldots = \overset{q}{\boldsymbol{v}} = 0.$$

In this case the general expressions (9) and (10) are certainly valid as well, and that for the scalar potential becomes:

$$\phi = \phi_0 = e\sum_{m=0}^{\infty}\frac{\partial^m R^{m-1}}{\partial \boldsymbol{R}^m}\frac{\boldsymbol{v}^m}{m!} = e\psi_{-1}. \tag{A.3}$$

On the other hand, for the uniform motion the potential ϕ can be expressed in terms of the simultaneous position and velocity of the particle in the form (16). Comparison of (A.3) with (16) shows that

$$\psi_{-1} = \{R[1-v^2+(\boldsymbol{n}\cdot\boldsymbol{v})^2]^{1/2}\}^{-1}. \tag{A.4}$$

and this expression remains the same in the general case for a motion with a varying velocity, because ψ_k are independent of acceleration.

With the help of (A.4) one can find the other ψ_k for $k = 0, 1, 2, \ldots$. Note, that from (A.2) one gets

$$\frac{\partial \psi_k}{\partial \boldsymbol{R}} = \frac{\partial \psi_{k-1}}{\partial \boldsymbol{v}}, \tag{A.5}$$

if the partial derivative with respect to $\boldsymbol{R}$ is taken at $\boldsymbol{v}$ = constant and vice versa. From (A.5) one sees that

$$\frac{\partial^m \psi_k}{\partial \boldsymbol{R}^m} = \frac{\partial^m \psi_{k-m}}{\partial \boldsymbol{v}^m}, \qquad \frac{\partial^{k+1} \psi_k}{\partial \boldsymbol{R}^{k+1}} = \frac{\partial^{k+1} \psi_{-1}}{\partial \boldsymbol{v}^{k+1}}.$$

Since the differentiation with respect to $\boldsymbol{v}$ does not change the dependence on $\boldsymbol{R}$, and since from (A.4) one sees that $\psi_{-1} \sim R^{-1}$, it is easy to conclude that $\psi_k \sim R^k$ and put

$$\psi_m = R^m \Phi_m(\boldsymbol{n}, \boldsymbol{v}), \tag{A.6}$$

where Φ_m are scalar functions of their arguments.

If one writes $\partial^m R^{m+k}/\partial \boldsymbol{R}^m$ in (A.2) as $\partial^m(R^2 R^{m+k-2})/\partial \boldsymbol{R}^m$, uses the formula for the mth derivative of the product of two functions R^2 and R^{m+k-2}, and notes that

$$\frac{\partial R^2}{\partial \boldsymbol{R}} = 2\boldsymbol{R}, \qquad \frac{\partial^2 R^2}{\partial \boldsymbol{R}^2} \Rightarrow \frac{\partial^2 R^2}{\partial R_i \partial R_j} = 2\delta_{ij}, \qquad \frac{\partial^m R^2}{\partial \boldsymbol{R}^m} = 0 \qquad (m > 2),$$

one arrives at the following equation for ψ_k:

$$\psi_k = R^2 \psi_{k-2} + 2(\boldsymbol{R} \cdot \boldsymbol{v})\psi_{k-1} + v^2 \psi_k.$$

Making use of (A.6), one obtains a recurrence relation for Φ_k:

$$(1 - v^2)\Phi_k = \Phi_{k-2} + 2(\boldsymbol{n} \cdot \boldsymbol{v})\Phi_{k-1}. \tag{A.7}$$

To find all Φ_k from this relation, one needs to know only two of them, say Φ_{-1} and Φ_{-2}. One of the functions Φ_{-1} has been defined in (A.4). The second can be found if one uses (A.5) for ψ_{-1}

$$\frac{\partial \psi_{-2}}{\partial \boldsymbol{v}} = \frac{1}{R^2} \frac{\partial \Phi_{-2}}{\partial \boldsymbol{v}} = \frac{\partial \psi_{-1}}{\partial \boldsymbol{R}}.$$

The solution to this simple differential equation, with the right-hand side defined by differentiation of (A.4) with respect to $\boldsymbol{R}$, is

$$\Phi_{-2} = 1 - \frac{\boldsymbol{n} \cdot \boldsymbol{v}}{[1 - v^2 + (\boldsymbol{n} \cdot \boldsymbol{v})^2]^{1/2}}. \tag{A.8}$$

The constant of integration here has been defined from the condition that, for a particle at rest, the potential should take its usual Coulomb form.

Now, it is useful to introduce instead of Φ_k some new functions M_k, defined by the relations

$$M_k = (1 - v^2)^{(k+2)/2}\Phi_k, \qquad \Phi_k = \gamma^{k+2} M_k,$$

where $\gamma = (1-v^2)^{-1/2}$, and a new variable

$$z = \frac{\boldsymbol{n}\,.\,\boldsymbol{v}}{(1-v^2)^{1/2}} = \gamma(\boldsymbol{n}\,.\,\boldsymbol{v}) = \boldsymbol{n}\,.\,\boldsymbol{u}.$$

In this notation (A.7) becomes

$$M_{k+2} = 2zM_{k+1} + M_k. \tag{A.9}$$

This recurrence relation is satisfied by

$$M_k = Q_m M_{k-m} + Q_{m-1} M_{k-m-1}, \tag{A.10}$$

where the polynomials Q_m are

$$Q_m = \sum_{i=0}^{E(m/2)} \frac{(m-i)!}{i!(m-2i)!}(2z)^{m-2i},$$

$E(m/2)$ being the integral part of $m/2$.

Straightforward differentiation convinces one that the Q_m can be expressed in terms of derivatives of γ:

$$Q_m = \frac{(1-v^2)[\boldsymbol{n}(1-v^2)^{1/2}]^m}{m!} \frac{\partial^m (1-v^2)^{-1}}{\partial \boldsymbol{v}^m} = \frac{\boldsymbol{n}^m}{\gamma^{m+2} m!} \frac{\partial^m \gamma^2}{\partial \boldsymbol{v}^m}. \tag{A.11}$$

Substitution for Q_m from (A.11) into (A.10) when $m = k$ gives

$$\begin{aligned} M_k &= Q_k M_0 + Q_{k-1} M_{-1}, \\ \Phi_k &= \gamma^{k+2} M_k = \frac{\gamma^{-2}\boldsymbol{n}^k}{k!} \frac{\partial^k \Phi_0}{\partial \boldsymbol{v}^k} \\ &= \frac{\boldsymbol{n}^k}{k!} \frac{\partial^k}{\partial \boldsymbol{v}^k}\left[\gamma^2 \left(1 + \frac{\boldsymbol{n}\,.\,\boldsymbol{u}}{[1+(\boldsymbol{n}\,.\,\boldsymbol{u})^2]^{1/2}}\right)\right], \end{aligned} \tag{A.12}$$

where M_0, M_{-1} have been found with the aid of (A.4), (A.8) and (A.7).

According to (A.12), the functions Φ_{k+2} and Φ_{k+1} can be expressed in terms of derivatives of Φ_k with respect to velocity $\boldsymbol{v}$, and the same is true for M_k. Therefore, the recurrence equation (A.9) can be written in the form of a differential equation for M_k as follows:

$$(1+z^2)\frac{\mathrm{d}^2 M_k}{\mathrm{d}z^2} + 3z\frac{\mathrm{d}M_k}{\mathrm{d}z} - k(k+2)M_k = 0. \tag{A.13}$$

A general solution to the equation is

$$M_k = A_1(1-y)\left(\frac{1-y}{1+y}\right)^{k/2} + A_2(1+y)\left(\frac{1+y}{1-y}\right)^{k/2}, \tag{A.14}$$

where a new variable $y = z/(1+z^2)^{1/2}$ has been introduced. When $k = -2, -1$, these M_k should coincide with the functions M_{-2} and M_{-1} that have been already found in (A.4) and (A.8). The comparison of (A.14) with these expressions shows that the constants A_1 and A_2 should be chosen to be $A_1 = 0$, $A_2 = 1$.

Finally, the expression for M_k, written in terms of the former variable z, is

$$M_k = \frac{[z+(1+z^2)^{1/2}]^{k+1}}{(1+z^2)^{1/2}}. \tag{A.15}$$

The formulae (A.12) and (A.15) coincide with (21), if one notes the connection between M_k and Φ_k.

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